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A Simple Proof of the Prime Number Theorem

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The Prime Number Theorem is proved using only properties of the Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ in its half plane of convergence, and simple facts of harmonic analysis.

1. INTRODUCTION

This note contains a proof of the Prime Number Theorem,

$$\pi(n) = \sum_{p \leqslant n} 1 \sim \frac{n}{\log n},$$

which uses only properties of the Dirichlet series

$$\sum_{n=1}^{\infty} n^{-s}$$

in its half-plane of convergence $\mathrm{Re}(s) > 1$, and simple facts of harmonic analysis. No use is made of the existence of an analytic or even continuous function extending $\sum_{n=1}^{\infty} n^{-s}$ to the pair of vertical half-lines $\mathrm{Re}(s) = 1$, $\mathrm{Im}(s) \neq 0$, nor of any Tauberian theorem (cf. Titchmarsh [1] for a standard proof).

2. FULL STATEMENT OF RESULT

Recall that $\Lambda(n) = \log p$ if n is a prime p or a power of p, and $\Lambda(n) = 0$ otherwise. One form of the Prime Number Theorem reads

$$\sum_{n=1}^{N} \Lambda(n) \sim N.$$

We shall actually obtain the following stronger result.

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THEOREM. $\sum_{n=1}^{N} \Lambda(n) = N + O(N(\log N)^{\delta-1/4})$, *for every* $\delta > 0$.

For $\mathrm{Re}(s) > 1$, set

$$f(s) = \sum_{n=1}^{\infty} (1 - \Lambda(n))\, n^{-s};$$

$-\sum_{n=1}^{\infty} \Lambda(n)\, n^{-s}$ is the logarithmic derivative of $\sum_{n=1}^{\infty} n^{-s}$ in $\mathrm{Re}(s) > 1$. The properties of $\sum_{n=1}^{\infty} n^{-s}$ in the open right half-plane $\mathrm{Re}(s) > 1$ that will be used are the following. There is a positive constant D such that

$$\int | f(1 + \epsilon + it)| \mid 1 + it \mid^{-p} dt \leqslant D, \tag{2.1}$$

and

$$\int | f'(1 + \epsilon + it)| \mid 1 + it \mid^{-p} dt \leqslant D, \tag{2.2}$$

for all p, $1 < p < 2$, uniformly in ϵ; here and elsewhere ϵ is an arbitrary positive real number; and an integration, without limits of integration indicated, is taken to mean integration over the entire real line R.

The two inequalities (2.1) and (2.2) are consequences of these facts; there is a positive real number t_0 such that $f(1 + \epsilon + it)$ and $f'(1 + \epsilon + it)$ are bounded when $|t| \leqslant t_0$, uniformly in ϵ; and

$$f(1 + \epsilon + it) = O(\log^9 t),$$

and

$$f'(1 + \epsilon + it) = O(\log^{18} t)$$

for $|t| > t_0$, uniformly in ϵ. These are stated, or directly implied by what is stated, in Titchmarsh [1, pp. 42–44], and their proofs depend primarily on the Euler summation formula and the inequality

$$\left(\sum_{n=1}^{\infty} n^{-1-\epsilon}\right)^3 \left| \sum_{n=1}^{\infty} n^{-1-\epsilon-it} \right|^4 \left| \sum_{n=1}^{\infty} n^{-1-\epsilon-2it} \right| \geqslant 1.$$

In addition we shall need the inequality

$$| 1 - \Lambda(n)| \leqslant \log n,$$

for $n > 1$.

The rest of this note is devoted to the proof of the theorem above.

3. A Family of Measures Associated with f

For each ϵ form the following complex-valued measure on R:

$$\mu_\epsilon = \sum_{n=1}^{\infty} (1 - \Lambda(n))\, n^{-1-\epsilon} \delta_{\log n}\,;$$

δ_c is the Dirac delta measure in R supported by c. The μ_ϵ have totally finite total variation: $\| \mu_\epsilon \| = \sum_{n=1}^{\infty} | 1 - \Lambda(n) | \, n^{-1-\epsilon}$. Moreover,

$$f(1 + \epsilon + it) = \int e^{-itx}\, d\mu_\epsilon(x), \tag{3.1}$$

and

$$\sum_{n=1}^{N} (1 - \Lambda(n))\, n^{-\epsilon} = \int_0^{\log N} e^x\, d\mu_\epsilon(x). \tag{3.2}$$

Our objective is to show that

$$\left| \int_0^{\log N} e^x\, d\mu_\epsilon(x) \right| = O(N(\log N)^{\delta-1/4}),$$

uniformly in ϵ. To do this we shall use a partition of unity on the interval $[0, \log N]$.

4. A Partition of Unity on $[0, \log N]$

Let M be a positive integer, which will be specified later, and set $b = M^{-1} \log N$. Let k be an integer, $0 \leqslant k \leqslant M$. Define g_k: $R \to R$ by $g_k(x) = 1 - b^{-1} | kb - x |$ for $(k - 1)b < x < (k + 1)b$, and $g_k(x) = 0$ for values of x elsewhere on R. For x between $(k - 1)b$ and $(k + 1)b$ the graph of g_k rises in a straight line from $((k - 1)b, 0)$ to $(kb, 1)$, and descends in a straight line from $(kb, 1)$ to $((k + 1)b, 0)$. Then,

$$\int_0^{\log N} e^x\, d\mu_\epsilon(x) = \sum_{k=0}^{M} \int g_k(x) e^x\, d\mu_\epsilon(x) - \int_{-b}^{0} g_0(x) e^x\, d\mu_\epsilon(x) \tag{4.1}$$
$$- \int_{Mb}^{(M+1)b} g_M(x) e^x\, d\mu_\epsilon(x).$$

5. Application of The Fourier Transform and The Fubini Theorem

The (unnormalized) Fourier transform of $g_k(x)\, e^x$ is everywhere equal to

$$t \mapsto b^{-1} e^{(1+it)(k-1)b} (e^{(1+it)b} - 1)^2 (1 + it)^{-2}.$$

Therefore, as $g_k(x)\, e^x$ is a continuous, absolutely integrable function of bounded variation, differentiable except at finitely many points, it is everywhere equal to the (inverse) Fourier transform of its Fourier transform above:

$$g_k(x)e^x = (2\pi)^{-1} \int b^{-1} e^{(1+it)(k-1)b}(e^{(1+it)b} - 1)^2(1 + it)^{-2}\, e^{-itx}\, dt,$$

for all x in R (cf. Titchmarsh [2, p. 42]).

Because

$$\begin{aligned}
&(2\pi)^{-1} \iint |\, b^{-1} e^{(1+it)(k-1)b}(e^{(1+it)b} - 1)^2(1 + it)^{-2}\, e^{-itx} \,|\, dt d \,|\, \mu_\epsilon \,|\, (x) \\
&\quad \leqslant (2\pi b)^{-1} \iint e^{(k-1)b}(e^b + 1)^2 \,|\, 1 + it \,|^{-2}\, dt d \,|\, \mu_\epsilon \,|\, (x) \\
&\quad \leqslant (2\pi b)^{-1}\, e^{(k-1)b}(e^b + 1)^2 \,\|\, \mu_\epsilon \,\| \int |\, 1 + it \,|^{-2}\, dt,
\end{aligned}$$

which is finite, Fubini's Theorem and (3.1) give

$$\begin{aligned}
&\int g_k(x)e^x\, d\mu_\epsilon(x) \\
&\quad = (2\pi b)^{-1} \int e^{(1+it)(k-1)b}(e^{(1+it)b} - 1)^2(1 + it)^{-2} f(1 + \epsilon + it)\, dt.
\end{aligned}$$

Therefore, from (4.1)

$$\begin{aligned}
&\int_0^{\log N} e^x\, d\mu_\epsilon(x) \\
&\quad = (2\pi b)^{-1} \int (N^{1+it} - e^{-(1+it)b})(e^{(1+it)b} - 1) f(1 + \epsilon + it)(1 + it)^{-2}\, dt \qquad (5.1) \\
&\qquad - \int_{-b}^{0} g_0(x)e^x\, d\mu_\epsilon(x) \qquad (5.2) \\
&\qquad - \int_{Mb}^{(M+1)b} g_M(x)e^x\, d\mu_\epsilon(x). \qquad (5.3)
\end{aligned}$$

The term (5.2) is equal to -1.

We now set $b = (\log N)^{p/4-6/4}$, so $M = (\log N)^{10/4-p/4}$.

6. Estimation of (5.3)

The term (5.3) equals

$$\sum_{N \leqslant n \leqslant Ne^b} g_M(\log n)\, n(1 - \Lambda(n))\, n^{-1-\epsilon}.$$

This is in absolute value less than or equal to

$$\sum_{N\leqslant n\leqslant Ne^b} |1 - \Lambda(n)|\, n^{-\epsilon} \leqslant \sum_{N\leqslant n\leqslant Ne^b} \log n$$
$$\leqslant N(\log(Ne^b))(e^b - 1 + N^{-1}).$$

Using the value assigned to b in Section 5, the last expression is, for large N, less than or equal to

$$N(\log(Ne^b))(2(\log N)^{p/4-6/4} + N^{-1}).$$

This is $O(N(\log N)^{\delta-1/4})$ uniformly in ϵ, where we put $\delta = (p-1)/4$.

7. Estimation of (5.1)

The term (5.1) is equal to

$$N(2\pi b)^{-1} \int N^{it}(e^{(1+it)b} - 1) f(1+\epsilon+it)(1+it)^{-2}\,dt \tag{7.1}$$

$$-(2\pi b)^{-1} \int e^{-(1+it)b}(e^{(1+it)b} - 1) f(1+\epsilon+it)(1+it)^{-2}\,dt. \tag{7.2}$$

The term (7.2) is by (2.1) equal to $O((\log N)^{6/4-p/4})$, uniformly in ϵ.

To estimate (7.1), we begin by integrating it by parts to get

$$-N(2\pi b \log N)^{-1} \int N^{it}(e^{(1+it)b} - 1) f'(1+\epsilon+it)(1+it)^{-2}\,dt \tag{7.3}$$

$$-N(2\pi \log N)^{-1} \int N^{it}e^{(1+it)b} f(1+\epsilon+it)(1+it)^{-2}\,dt \tag{7.4}$$

$$+N(\pi b \log N)^{-1} \int N^{it}(e^{(1+it)b} - 1) f(1+\epsilon+it)(1+it)^{-3}\,dt. \tag{7.5}$$

The term (7.4) is $O(N(\log N)^{\delta-1/4})$, uniformly in ϵ, by (2.1).

Now take N large. Let $T = (\log N)^{1/2}$, so $b = T^{p/2-3}$. To estimate (7.3), we partition the path of integration R into three subintervals, $(-\infty, -T]$, $(-T, T)$, $[T, +\infty)$, and treat the integrations of the integrand in (7.3) over these intervals separately.

The parts of (7.3) corresponding to the integrations over $(-\infty, -T]$ and $[T, +\infty)$ are, for N large, each bounded in absolute value by

$$N(2\pi T^{p/2-3}T^2)^{-1} \int 3T^{p-2}\, |f'(1+\epsilon+it)|\, |1+it|^{-p}\,dt$$
$$\leqslant 3(2\pi)^{-1} ND(\log N)^{p/4-1/2},$$

which is $O(N(\log N)^{\delta-1/4})$ uniformly in ϵ.

The part of (7.3) corresponding to the integration over $(-T, T)$ is bounded in absolute value by

$$(2\pi)^{-1} N(\log N)^{1/2-p/4} \max_{-T\leqslant t\leqslant T} | e^{(1+it)b} - 1 |$$
$$\times \int |f'(1 + \epsilon + it)| \, | 1 + it |^{-2} \, dt.$$

Because for N large and $-T \leqslant t \leqslant T$ we have $| e^{(1+it)b} - 1 | \leqslant 2Tb = 2(\log N)^{p/4-1}$, and because of (2.2), the term above is less than or equal to

$$\begin{aligned} N(2\pi)^{-1} D(\log N)^{1/2-p/4} \, 2(\log N)^{p/4-1} &= N\pi^{-1}D(\log N)^{-1/2} \\ &= O(N(\log N)^{\delta-1/4}) \end{aligned}$$

uniformly in ϵ.

The term (7.5) is shown to be $O(N(\log N)^{\delta-1/4})$ uniformly in ϵ, by a similar argument.

References

1. E. C. Titchmarsh, "The Theory of the Riemann Zeta-Function," Oxford University Press, Oxford, 1951.
2. E. C. Titchmarsh, "Theory of Fourier Integrals," Oxford University Press, Oxford, 1937.